

# Magnetic moment of relativistic fermions

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In the paper a new class of exact localized solutions of Dirac's equation in the field of a circularly polarized electromagnetic wave and a constant magnetic field is presented. These solutions possess unusual properties and are applicable only to relativistic fermions. The problem of the magnetic resonance is considered in the framework of the classical theory of fields. It is shown that interpretation of the magnetic resonance for relativistic fermions must be changed. Numerical examples of parameters of the electromagnetic wave, constant magnetic field and the localization length scale for real measurements are presented.

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## INTRODUCTION

It is well known that exact solutions of quantum mechanics equations are of fundamental importance in physics. However, only few such solutions are known. We present a new class of exact localized solutions of the Dirac equation in the field of traveling circularly polarized electromagnetic wave and a constant magnetic field applied along the wave propagation.

It also is well known that the problem of the magnetic moment is of special interest in physics. Its contemporary value is deduced in the framework of quantum electrodynamics [1] and for electron have been confirmed with the amazing accuracy in a variety of experiments [2]. However, these measurements correspond to not large velocities and not very strong magnetic fields. The theoretical basis for that is the Pauli equation and the magnetic resonance interpretation as a flop of spin under action of a constant and oscillating magnetic field [3]. In the Pauli equation the operator of the magnetic moment is proportional to that of spin. In the relativistic case this dependence is more complicated.

In the paper we consider this problem and find an exact analytical solution for this case. Since operators of energy, momentum and spin in a variable magnetic field do not commute with the Hamiltonian, we must consider average values of these operators. In fact this consideration is equivalent to that in the framework of the classical theory of fields [4]. We show that in the relativistic case the magnetic resonance is a flop of the magnetic moment but not spin.

We consider numerical examples of presented solutions applicable to experimental testing.

## SOLUTIONS OF DIRAC'S EQUATION

Consider Dirac's equation

$$i\hbar\frac{\partial}{\partial t}\Psi = c\alpha(\mathbf{p} - \frac{e}{c}\mathbf{A})\Psi + \beta mc^2\Psi = 0 \quad (1)$$

in the field of a traveling circularly polarized electromagnetic wave and a constant magnetic field  $H_z$  directed along the  $z$ -axis. Such a field corresponds to the potential

$$A_x = -\frac{1}{2}H_z y + \frac{1}{k}H \cos(\Omega t - kz), \quad (2)$$

$$A_y = \frac{1}{2}H_z x + \frac{1}{k}H \sin(\Omega t - kz), \quad (3)$$

where  $k = \varepsilon\Omega/c$  is the propagation constant,  $\Omega$  is the frequency, the sign change of  $\Omega$  corresponds to the opposite polarization, values  $\varepsilon = 1$  and  $\varepsilon = -1$  are used when the wave propagates along the  $z$ -axis and opposite direction respectively,  $c$  is the speed of light,  $\alpha_k, \beta$  are Dirac's matrices,  $H$  is the amplitude of this wave. It is well known that the amplitudes of the electric and magnetic fields in the plane wave coincide. These fields are an integral part of the wave and the influence of the electric and magnetic field cannot be considered separately in the given case.

The transition into a rotating frame is essential for the modulation by a rotating electromagnetic field since in such a frame stationary solutions are possible. It is well known that the transformation  $\Psi' = \exp(\frac{1}{2}\alpha_1\alpha_2\Phi)\Psi$  describes a spinor in the frame turned around the  $z$ -axis through an angle  $\Phi$ . If  $\Phi = \Omega t$ , then it describes a spinor in the rotating frame. In accordance with this, an appropriate coordinate transformation is also necessary. Below we use the tilde for coordinates in the rotating frame

$$\tilde{x} = r \cos \tilde{\varphi}, \quad \tilde{y} = r \sin \tilde{\varphi}, \quad (4)$$

$$\tilde{\varphi} = \varphi - \Omega t + kz, \quad \tilde{t} = t, \quad \tilde{z} = z. \quad (5)$$

This transformation is identified as a "Galilean transformation for rotating frames" because time in the initial and rotating frame is the same.

We use constants  $E$  and  $p$  as "energy" and "momentum along the  $z$ -axis" for stationary states in the rotating frame. Once these states have been found the wave function as well as coordinates are translated back into the initial (non-rotating) frame.

In the initial frame Eq. (7) has exact solutions local-

ized perpendicularly to the  $z$ -axis [5]

$$\Psi = \exp[-i\frac{E}{\hbar}t + i\frac{p}{\hbar}z - \frac{1}{2}\alpha_1\alpha_2(\Omega t - kz) + D]\psi, \quad (6)$$

$$D = -\frac{1}{2}d(\tilde{x}^2 + \tilde{y}^2) + d_1\tilde{x} + d_2\tilde{y}, \quad (7)$$

$\psi$  is a spinor polynomial in  $\tilde{x}$  and  $\tilde{y}$ . If  $\psi$  is a constant spinor then solution (6) describes the "ground state". Below, as a simplest example, we investigate properties of this state.

Localized solutions (6) exist if the parameter  $d$  is positive and defined by the equality

$$d^2 = \frac{e^2}{4\hbar^2 c^2} H_z^2. \quad (8)$$

In accordance with Eq. (8), two types of solutions are possible. We denote them as  $\psi_-$  for  $eH_z < 0$  and  $\psi_+$  for  $eH_z > 0$ . These normalized spinors of the ground state have the form

$$\psi_- = N_- \begin{pmatrix} h\mathcal{E} \\ -\varepsilon(\mathcal{E}+1)(\mathcal{E}-\mathcal{E}_0) \\ \varepsilon h\mathcal{E} \\ -(\mathcal{E}-1)(\mathcal{E}-\mathcal{E}_0) \end{pmatrix}, \quad (9)$$

$$\psi_+ = N_+ \begin{pmatrix} (\mathcal{E}+1)(\mathcal{E}+\mathcal{E}_0) \\ \varepsilon\mathcal{E}h \\ -\varepsilon(\mathcal{E}-1)(\mathcal{E}+\mathcal{E}_0) \\ -\mathcal{E}h \end{pmatrix}, \quad (10)$$

where  $N_{\mp}$  is defined by the normalization condition  $\int \Psi^* \Psi dxdy = 1$ .

$$N_{\mp} = \frac{\sqrt{d/2\pi} \exp(-d_2^2/2d)}{\sqrt{(\mathcal{E}^2+1)(\mathcal{E}\mp\mathcal{E}_0)^2 + h^2\mathcal{E}^2}}, \quad (11)$$

$$d_1 = \mp id_2, \quad d_2 = \frac{\mathcal{E}_0 mch}{2\hbar(\mathcal{E}\mp\mathcal{E}_0)}. \quad (12)$$

The upper and lower sign before a parameter corresponds to solutions with negative and positive  $eH_z$  respectively.

Eigenvalues of "energy in the rotating frame"  $E$  are defined with help of a normalized parameter  $\mathcal{E} \equiv -(E - \varepsilon pc)/mc^2$ . This parameter obeys the equation

$$\mathcal{E}^3 + (\mp\mathcal{E}_0 + \Lambda_{\mp})\mathcal{E}^2 - (1 \pm \mathcal{E}_0\Lambda_{\mp} + h^2)\mathcal{E} \pm \mathcal{E}_0 = 0, \quad (13)$$

where

$$\mathcal{E}_0 = \frac{2\hbar d}{\Omega m}, \quad \Lambda_{\mp} = \frac{2\varepsilon pc \mp \hbar\Omega}{mc^2}, \quad h = \frac{e}{kmc^2} H. \quad (14)$$

Obviously, wave functions (9) and (10) cannot be presented as a small and large two-component spinor. It means that the difference  $E^2 - m^2c^2$  cannot be small.

From this an important conclusion follows: *these solutions correspond only to the relativistic case*. Another distinguish feature is an algebraic equation of the third order (13) for eigenvalues  $\mathcal{E}_k$  and correspondingly  $E_k$ ,  $k = 1, 2, 3$ .

The wave function  $\psi_{\mp}$  depends on four independent normalized parameters  $\mathcal{E}_0, \Lambda_{\mp}, d, h$  which are used below rather than  $\Omega, p, H_z, H$ .

## AVERAGE ENERGY AND MOMENTUM

It should be emphasized that in the initial frame states are not stationary, operators of energy and momentum don't commute with the Hamiltonian. Therefore average values of operators must be used. In the initial frame average energy  $\bar{E} \equiv i\hbar \int \Psi^* \frac{\partial}{\partial t} \Psi dxdy$  and momentum  $\bar{\mathbf{p}} \equiv -i\hbar \int \Psi^* \nabla \Psi dxdy$  are as follows

$$\bar{E} = E \mp \zeta_{\mp} \pm \hbar\Omega \frac{d_2^2}{d}, \quad (15)$$

$$\bar{p}_z = p \mp \frac{\varepsilon}{c} \zeta_{\mp} \pm \hbar\Omega \frac{\varepsilon d_2^2}{cd}, \quad (16)$$

$$\bar{p}_x = \mp \hbar d_2 \cos(\Omega t - kz), \quad (17)$$

$$\bar{p}_y = \pm i\hbar d_2 \sin(\Omega t - kz), \quad (18)$$

where

$$\zeta_{\mp} = \frac{\hbar\Omega}{2} \frac{(\mathcal{E}^2+1)(\mathcal{E}\mp\mathcal{E}_0)^2 - h^2\mathcal{E}^2}{(\mathcal{E}^2+1)(\mathcal{E}\mp\mathcal{E}_0)^2 + h^2\mathcal{E}^2}. \quad (19)$$

For states (9) and (10) the uncertainty principle results in the equality

$$\Delta x \Delta p_x = \frac{1}{2} \hbar$$

where  $\Delta x = \sqrt{1/2d}$  and  $\Delta p_x = \hbar\sqrt{d/2}$  are standard deviations of  $x$  and momentum  $p_x$ . All that is valid also for deviations of  $y$  and momentum  $p_y$ .

## THE MAGNETIC MOMENT

It is well known that the magnetic moment plays a principal role in the interaction of fermions with a magnetic/electromagnetic field. For relativistic fermions described here the magnetic moment is determined as the first derivative of Dirac's Lagrangian with respect to the magnetic field. Since solutions (6) are localized perpendicularly to the  $z$ -axis, it is appropriate to consider only the  $z$ -component of the magnetic moment. This component is

$$\mu_z = \frac{e}{2} \int (y\Psi^* \alpha_1 \Psi - x\Psi^* \alpha_2 \Psi) dxdy. \quad (20)$$

This definition coincides with that in classical theory of fields [4]. Below we study this quantity contrary to the

classical consideration where a constant in the the Pauli equation is used instead  $\mu_z$ .

In non-relativistic case a rotating electromagnetic field gives rise electron pairs with spins rotating in opposite direction [5]. The wave function of electron is formed from two wave functions corresponding two stationary states in the rotating frame with a certain difference between energy levels. In the given case the wave function also has to be constructed from two functions

$$\Psi = C_1 \Psi_1 + C_2 \Psi_2, \quad (21)$$

where  $C_k$  is the normalization coefficient

$$|C_1|^2 + |C_2|^2 = 1.$$

The wave function (6) depends on four parameters  $\mathcal{E}_0, \Lambda_{\mp}, d, h$  and two roots of Eq. (13). For the study of magnetic moment properties it is convenient to introduce two new parameters

$$\Pi = \mathcal{E}_1 \mathcal{E}_2, \quad \eta = \mathcal{E}_1 + \mathcal{E}_2. \quad (22)$$

Then from Eq. (13) one follows

$$h^2 = -\frac{\Pi + 1}{\Pi} (1 \mp \mathcal{E}_0 \eta + \mathcal{E}_0^2), \quad \Lambda_{\mp} = \pm \mathcal{E}_0 - \eta \mp \frac{1}{\Pi} \mathcal{E}_0 \quad (23)$$

Four new variable parameters  $\mathcal{E}_0, \Pi, \eta, d$  can be considered as independent. With these parameters there is not need to refer to Eq. (13).

Substitute (21) in Eq. (20). The integration in (20) results in a exact expression. This expression consists of a time-independent and oscillating part. The constant part may be reduced to zero if  $\Pi < 0$ , i.e., if  $\mathcal{E}_1$  and  $\mathcal{E}_2$  have opposite signs. The variable part oscillates at the frequency

$$\frac{E_1 - E_2}{\hbar}. \quad (24)$$

The dependence on  $d$  may be extracted from the variable part:

$$\frac{1}{d} \exp\left[-\frac{(d'_2 - d''_2)^2}{2d}\right], \quad (25)$$

where  $d'_2$  and  $d''_2$  are independent on  $d$  (12) and correspond to states with  $\mathcal{E}_1$  and  $\mathcal{E}_2$  respectively.

## THE MAGNETIC RESONANCE

It is well known that in non-relativistic case the magnetic resonance occurs at a value of the longitudinal magnetic field corresponding to the spin oscillation amplitude maximum. At this value the amplitude becomes constant.

Such a maximization in relativistic case corresponds to an extremum of the function (25) in respect to  $d$ . At the extremum point

$$2d = (d'_2 - d''_2)^2. \quad (26)$$

With help of this equality the parameter  $d$  may be excluded from consideration. However, in contrast to the classical case, the amplitude of the magnetic moment oscillation remains dependent on three parameters  $\Pi, \mathcal{E}_0, \eta$ . The change of these parameters allows to vary the magnetic moment over a wide range. The exact expression for the oscillation amplitude  $A_{\mp}$  at the standard classical condition  $|C_1|^2 = |C_2|^2 = \frac{1}{2}$  takes the form

$$A_- = \mu_B \frac{4R_-}{\mathcal{E}_0 \exp(1)} [-2\Pi + 2\mathcal{E}_0 \eta - 2\mathcal{E}_0^2], \quad (27)$$

$$A_+ = \mu_B \frac{4R_+}{\mathcal{E}_0 \exp(1)} [2\Pi - 2\eta \mathcal{E}_0 - \eta^2 - 2\mathcal{E}_0^2], \quad (28)$$

$$R_{\mp} = \sqrt{\frac{-\Pi^3 (\Pi \mp \mathcal{E}_0 \eta + \mathcal{E}_0^2)}{(\Pi^3 \pm \Pi \eta \mathcal{E}_0 + \mathcal{E}_0^2)(\eta^2 - 4\Pi)^3}}. \quad (29)$$

$\mu_B$  is the Bohr magneton.

Other pair of wave functions may be considered analogously. But the maximization is possible only for one pair of wave functions.

It is easy to show that the magnetic moment oscillation frequency (24), in contrast to the classical case, depends on the fermion mass (and  $\mathcal{E}_0, \Pi, \eta$  or non-normalized parameters  $\Omega, p, h$ ). Therefore, measurements of this frequency probably may be turned into the precision measurement of fermion masses, provided an experimental technique will be developed for the estimate of the number of oscillations and fermion time-of-flight. Particular cases may be calculated with help of exact expressions for  $A_{\mp}$ .

An consideration of the average spin  $s_3 = -\frac{i}{2} \hbar \int \Psi^* \alpha_1 \alpha_2 \Psi dx dy$  shows that  $s_3$  also consists of a constant and oscillating part. The time-dependent part oscillates with the same frequency (24), however, the amplitude of oscillations has no an extremum by varying  $H_z$  and monotonically decreases to zero at the change of  $H_z$  from infinity to zero. From this the second important conclusion follows: *in the relativistic case the magnetic resonance is the flop of the magnetic moment but not spin.*

## SOME EXAMPLES

In this Section we consider particular values of  $H_z, \Omega, H$  for which a real testing of presented solutions is possible. For definiteness we consider examples for electron.

We start from such a characteristic feature as the localization length scale  $l$ . This scale perpendicular to the

$z$ -axis is defined by the parameter  $d$  (8)

$$l \approx 2\sqrt{\frac{1}{d}} = 2\sqrt{\left|\frac{2\hbar c}{eH_z}\right|}. \quad (30)$$

The frequency  $\omega = \Omega/2\pi$  may be found from particular values of  $\mathcal{E}_0$  (14). The first example illustrates the  $l$ - and  $\omega$ - dependence of  $H_z$  from relatively weak to strong values. The magnetic field of the order of  $40T$  is used as an upper limit of the man-made magnetic field producing the scale lower limit. In this example  $\mathcal{E}_0 = 1$ .

|              |                      |                      |                      |
|--------------|----------------------|----------------------|----------------------|
| $H_z(G)$     | 1                    | $10^3$               | $4 \cdot 10^5$       |
| $l(cm)$      | $7.26 \cdot 10^{-4}$ | $2.29 \cdot 10^{-5}$ | $1.15 \cdot 10^{-6}$ |
| $\omega(Hz)$ | $2.80 \cdot 10^6$    | $2.80 \cdot 10^9$    | $1.12 \cdot 10^{12}$ |

Table 1

These dependences are not connected with the magnetic resonance.

The magnetic resonance places the constraint on  $d$  (26). The second example illustrates  $\omega$ - and  $h$ - dependence of  $H_z$  in the condition of the magnetic resonance and at the condition

$$|\mathcal{E}_0| \ll 1. \quad (31)$$

This condition enables to simplify calculations. Moreover, the inequality  $|\hbar\Omega/mc^2| \ll 1$  is always holds. Therefore, for relativistic fermions  $\Lambda_{\mp} = 2\varepsilon p/mc$ . In this example  $|\mathcal{E}_0| = 0.4117 \cdot 10^{-2}$  is chosen so that  $\omega$  corresponds to the frequency of a powerful Nd:YAG laser with the wavelength of 1.064 micron.

|              |                      |                      |                      |
|--------------|----------------------|----------------------|----------------------|
| $H_z(G)$     | 1                    | $10^3$               | $4 \cdot 10^5$       |
| $\omega(Hz)$ | $7.04 \cdot 10^7$    | $7.04 \cdot 10^{10}$ | $2.82 \cdot 10^{14}$ |
| $hP$         | $3.66 \cdot 10^{-5}$ | $1.17 \cdot 10^{-3}$ | $2.31 \cdot 10^{-2}$ |

Table 2

where  $P = \sqrt{p^2/m^2c^2 + 1}$ . A corresponding value of  $H$  may be found from the relation  $H = \varepsilon h H_z / \mathcal{E}_0$ .

The laser radiation may be transformed by means of optics in a beam with the cross-section of the order of

few microns. This cross-section may be much more than  $l$ . The energy density by this transformation increases by a factor of the order of  $10^8$  times. Such data may be used for flows of relativistic fermions of a very small cross-section driven by a powerful laser beam.

Above examples far not exhaust all possibilities for experiments.

## CONCLUSION

We have considered the new class of exact solutions of the Dirac equation corresponding to relativistic fermions. Practical numerical examples presented in the paper can describe, in particular, flows of relativistic fermions of a very small cross section driven by a powerful laser beam in accelerators and colliders. We have found the exact analytical solution for the problem of the magnetic resonance and shown that in relativistic case the magnetic resonance is a flop of the magnetic moment but not spin. Perhaps such a resonance may be turned into precision measurements of fermion masses. Considered solutions may be an useful and powerful instrument as for high energy physics as for astrophysics, in particular, for the description of processes near neutron stars and magnetars.

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